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Instabilities in liquid metals controlled by constant magnetic field—Part II: horizontal magnetic field

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Abstract

We investigate the stabilizing effects of a constant horizontal magnetic field on the flow in a heated planar liquid metal layer. The magnetic field is either transverse or longitudinal with respect to the imposed temperature gradient driving the buoyant shear flow. Two main instabilities are triggered in such flows: stationary transverse instabilities and oscillatory longitudinal instabilities. Each magnetic field is unable to stabilize the instabilities which develop in a plane perpendicular to its direction, so that the transverse field only affects the longitudinal instabilities and the longitudinal field the transverse instabilities. The results obtained by an approximate analytical linear stability analysis reveal a similar behaviour in both cases: small variation of the thresholds (critical Grashof number, Gr_c) and of the wavelengths at small *Ha* (*Ha* is the Hartmann number proportional to the intensity of the magnetic field) before an increase leading to asymptotic linear variations at large *Ha*. Comparisons with the results obtained in the case of a vertical magnetic field [S. Kaddeche et al., J. Crystal Growth, Part 1] show that the horizontal fields generate less efficient stabilizing effects. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Progress in material processing technologies is in part connected to the better control of fluid flow and heat and mass transport in the liquid phase. Towards this aim, the use of a magnetic field appears to be an interesting tool, both to brake the flow and to damp the oscillations. Following the experimental evidence that static magnetic field can lead to the suppression of the temperature fluctuations [2–4], it is proposed to analyse more carefully the influence of a magnetic field on the development of instabilities in a bounded liquid metal layer where the flow is generated by a horizontal temperature gradient. In the companion

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paper [1], the influence of a vertical magnetic field was studied, and it was shown that a vertical field strongly stabilizes both longitudinal and transverse instabilities which prevail in such buoyant flows of conducting materials.

In the present work, we focus on the influence of a horizontal field which can be either transverse or longitudinal with respect to the imposed temperature gradient driving the buoyant flow. Our objectives are to investigate the stabilizing efficiency of such horizontal magnetic fields and to compare with the results obtained in the case of a vertical field, thus allowing to see the influence of the magnetic field direction. It is well known that the braking effect of a magnetic field on steady convection greatly depends on its direction [5,6]; it will be of interest to see if the same behaviour will be observed for the stabilization effects. The influence of the magnetic field direction on the threshold parameters at onset of instability was analysed by Touihri et al. [7] for the vertical cylinder heated from below and by Priede and Gerbeth [8,9] for thermocapillary flows in an infinite layer. Both studies show that a vertical magnetic field has a more significant stabilizing effect, due to the fact that for the horizontal field the instability will develop in a plane perpendicular to the field direction where it will be less damped.

The analysis of the influence of the horizontal magnetic field on the longitudinal or transverse instabilities is conducted through approximate analytical stability studies similar to those used for the vertical field [1]. The predictions are compared to the experimental results of Hurle et al. [2], and whenever possible, the results are expressed through scaling laws of interest for the crystal grower. A global synthesis of the results obtained for the different orientations of the magnetic field is finally proposed.

2. Governing equations and laminar basic flow

We consider the flow induced by a horizontal temperature gradient in an infinite fluid layer subject to an external uniform magnetic field. Details concerning the model and the governing equations can be found in the companion paper [1]. We have only to emphasize that the direction of the external uniform magnetic field is now horizontal, either along the x direction (longitudinal field) or along the y direction (transverse field) (see Fig. 1). In both situations, a parallel flow solution exists ($\vec{V} = (U_0(z), 0, 0)$) which is governed by a set of two equations,

$$\frac{d^3 U_0}{dz^3} - Gr = 0, \tag{1}$$

$$\frac{d^2 T_0}{dz^2} = Pr\, U_0, \tag{2}$$

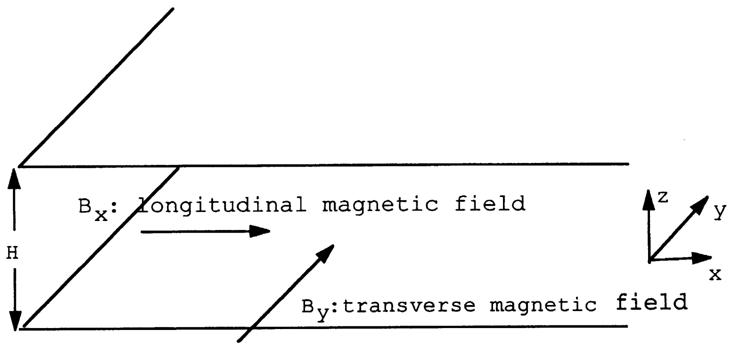

Fig. 1. Studied configuration.

which do not involve magnetic terms. This is due to the fact that the induced current, $\vec{j} = -\vec{\nabla}\phi + \vec{V}_0 \times \vec{B}_0$, is zero in both cases, for the longitudinal field because the direct induced current $\vec{V}_0 \times \vec{B}_0$ is zero, and for the transverse field because the potential induced current annihilates the direct induced current ($-\vec{\nabla}\phi + \vec{V}_0 \times \vec{B}_0 = 0$) [3]. Eqs. (1)–(2) give the following solutions which constitute the basic state of our linear stability analysis:

$$U_0(z) = \frac{Gr}{6}\left(z^3 - \frac{z}{4}\right), \tag{3}$$

$$T_0(x,z) = x + \frac{Gr\,Pr}{6}\left(\frac{z^5}{20} - \frac{z^3}{24} + \frac{7z}{960}\right) \text{ for thermally conducting boundaries,} \tag{4}$$

$$T_0(x,z) = x + \frac{Gr\,Pr}{6}\left(\frac{z^5}{20} - \frac{z^3}{24} + \frac{z}{64}\right) \text{ for thermally insulating boundaries.} \tag{5}$$

3. Linear stability analysis for the oscillatory longitudinal instabilities

The shear flow described by Eqs. (3)–(5) is known to be unstable at sufficiently large values of the Grashof number, either with respect to transverse instabilities or to longitudinal instabilities (small Pr domain). In this section, we focus on the oscillatory longitudinal instabilities defined by a zero wave number along x and corresponding to a three-dimensional mode which consists of rolls with the axis parallel to the basic flow. The stabilization of these longitudinal instabilities by horizontal transverse or longitudinal magnetic field is considered. In fact, the stabilization will only be effective if the direction of the field is transverse to the basic flow ($\vec{e}_{B_0} = \vec{e}_y$). Indeed, if the magnetic field is longitudinal ($\vec{e}_{B_0} = \vec{e}_x$), it is perpendicular to the plane of the longitudinal instabilities (yOz), so that there is no Lorentz force effect because the induced electric current is zero due to electric potential effects. Consequently, for this longitudinal field, the thresholds will be independent of Ha and correspond to the values obtained without magnetic field ($Ha = 0$) [1].

The approximate analytical linear stability study of the stabilization of the longitudinal instabilities by a horizontal transverse magnetic field will follow the approach performed in the companion paper [1] dealing with a vertical magnetic field. Nevertheless, rather than explicitly dealing with the induced electric potential perturbation, an approximation valid on the whole domain of Ha will be proposed. The governing linear equations for the perturbation in the present case are similar to those given in Ref. [1] (Eqs. (11)–(13)) except for the magnetic terms

$$\theta_t + uT_{0x} + \psi_y T_{0z} = \frac{1}{Pr}\Delta\theta, \tag{6}$$

$$u_t + \psi_y U_{0z} = \Delta u - Ha^2(u - \phi_z), \tag{7}$$

$$\Delta\psi_t = Gr\,\theta_y + \Delta^2\psi - Ha^2\psi_{yy}. \tag{8}$$

In these perturbation equations, u is the x component of velocity, θ the temperature, ψ the stream function in the (yOz) plane, ϕ the induced electric potential, and the operator Δ is defined as $\Delta = \partial^2/\partial y^2 + \partial^2/\partial z^2$. The treatment of these equations will be done as in Ref. [1]: a single equation in terms of ψ will be obtained, for which the solution will be of the form:

$$\psi = e^{\sigma t}\sin(\ell y)\cos(\pi z), \tag{9}$$

where ℓ and π are the wave numbers according to the y and z directions, respectively, and σ is a complex growth rate. Concerning the induced electric potential, we can remark that it acts only through the term ϕ_z

in the x component of the Lorentz body force, namely $F_x = -Ha^2(u - \phi_z)$. In order to estimate F_x, we may consider the electric current continuity equation written as follows:

$$\nabla^2\phi = \vec{e}_{B_0} \cdot \left(\vec{\nabla} \times \vec{V}\right). \tag{10}$$

In our case (transverse magnetic field, longitudinal instability), this equation becomes

$$\phi_{yy} + \phi_{zz} = u_z. \tag{11}$$

According to the type of developments given in (9), we have $\phi_{yy} = -\ell^2\phi$ and $\phi_{zz} = -\pi^2\phi$. For $Ha = 0$, we have $\ell^2/\pi^2 \propto 0.288$ (calculated from $\lambda = 2\pi/\ell \propto 3.728$ given by Eq. (29) in Ref. [1]). For $Ha \neq 0$, it will be found that ℓ will decrease with the increase of Ha, reaching an asymptotic variation $\ell \propto Ha^{-1}$ in the high Hartmann number limit. As a consequence, whatever Ha is, $\ell^2/\pi^2 \ll 1$, and then $|\phi_{yy}| \ll |\phi_{zz}|$. Eq. (11) will then give $\phi_{zz} \propto u_z$, and for cavities with electrically insulating boundaries, one can deduce that $\phi_z \propto u$, and so $F_x \propto 0$. According to this last result, the stability problem becomes independent of the induced electric potential ϕ.

As in Ref. [1] (but a little more simply as the potential is no more present in Eq. (7)), the stability equations of the problem under the given approximations lead to an expression of the Grashof number as a function of k ($k^2 = \ell^2 + \pi^2$) and Pr, namely

$$Gr^2(k) = -\frac{(1+Pr)k^2[2k^4 + Ha^2(k^2-\pi^2)][(1+Pr)k^4 + Ha^2\,Pr(k^2-\pi^2)]}{Pr^2(k^2-\pi^2)[k^2\bar{v}_z + k^4\bar{\tau}_z(1+Pr) + \bar{\tau}_z\,Ha^2(k^2-\pi^2)Pr]}, \tag{12}$$

where $\bar{v}_z$ and $\bar{\tau}_z$ are appropriately weighted mean values derived from U_{0z} and T_{0z} and given in Appendix A. The computation of the critical Grashof number Gr_c is achieved by minimizing the $Gr(k)$ function given by Eq. (12) with respect to k, which also gives k_c from which the critical wave number of the perturbation, l_c, is calculated. In Fig. 2, the neutral stability curves giving the variation of Gr_c as a function of Ha for different Pr, show the general stabilizing effect of the horizontal transverse magnetic field on the

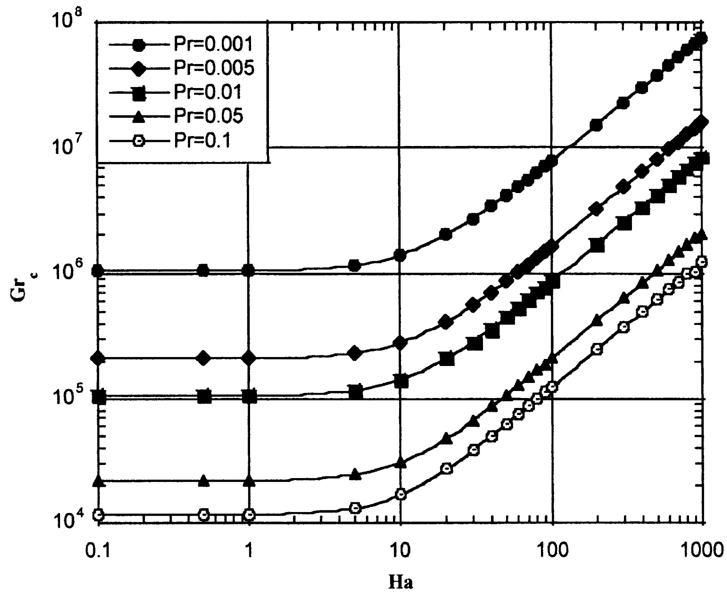

Fig. 2. Variation of Gr_c versus Ha for the oscillatory longitudinal instabilities under a horizontal transverse magnetic field (different values of the Prandtl number).

longitudinal instabilities. For moderate Hartmann numbers ($Ha < 1$), Gr_c remains almost constant indicating that the stabilizing effect in that case is not too significant. The increase becomes significant beyond $Ha = 10$, and, for higher Ha ($Ha > 100$), one can notice that the critical Grashof number Gr_c is a linear function of Ha in the considered range of Pr. This indicates a clear linear scaling of Gr_c at large Ha, i.e., $Gr_c \propto Ha$. Concerning the variation of the critical Grashof number versus the Prandtl number, a similar behaviour than that described in the companion paper dealing with the case of the vertical magnetic field [1] is observed. In particular, for low Prandtl numbers, the critical Grashof number Gr_c is found as a linear function of Pr^{-1}, i.e., $Gr_c \propto Pr^{-1}$. As a consequence, the relevant scaling law governing the variations of Gr_c according to the Prandtl and the Hartmann numbers, can be written for large Ha and low Pr as $Gr_c \propto Ha\, Pr^{-1}$.

These analytical predictions should be compared to the experimental results of Hurle et al. [2] who have studied the influence of a horizontal transverse magnetic field on the critical Rayleigh number for the onset of oscillatory instabilities ($Ra_c = Gr_c\, Pr$) in a top free cavity (boat) heated from the end sides and containing molten gallium ($Pr = 0.02$). We compare with the results corresponding to the 3.4 cm long boat because they are the most representative of the assumption of infinite horizontal extensions we have made in our analytical approach. The experimental and the analytical results are presented in Fig. 3 as Ra_c as a function of Ha^2. We can first notice that, in the range $0 \leqslant Ha^2 \leqslant 350$ investigated by Hurle et al. [2], both experimental and analytical results can be globally fitted by a Ha^2 law. Moreover, we can see that the values of the critical Rayleigh number compare favorably within a factor 3. The discrepancy is in part due to the fact that the analytical approach does not account for certain significant features such as the finite spatial dimensions of the boat.

As shown in Fig. 4, the wavelengths of the perturbations ($\lambda_c = 2\pi/\ell_c$) have a similar behaviour with Ha as the critical Grashof number, with a very small increase for small Ha, and a significant increase beyond $Ha = 10$ which tends towards an asymptotic linear variation for large Ha ($\lambda_c \propto Ha$). The oscillation frequencies f_c given in Fig. 5 behave differently. Beyond the domain of almost constant values at small Ha, there is still an increase in the intermediate domain of Ha, but this increase levels off quickly because the

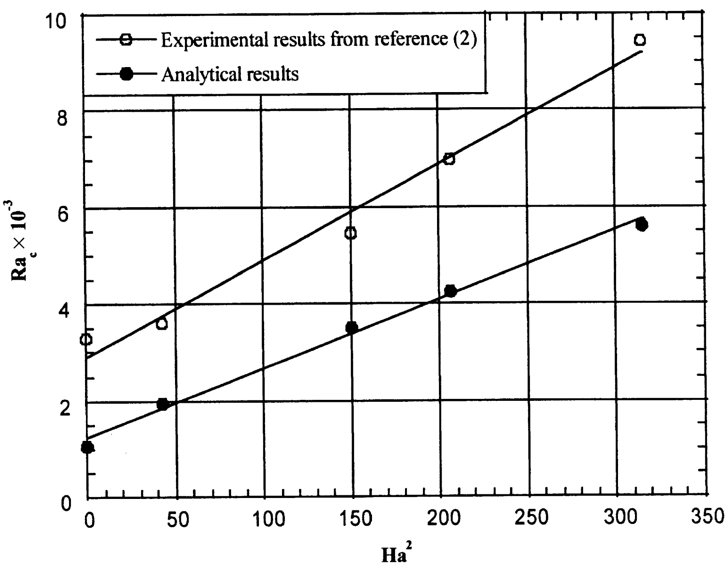

Fig. 3. Variation of the thresholds expressed by Ra_c versus Ha^2 for oscillatory instabilities under a horizontal transverse magnetic field. Comparison between analytical and experimental results [2].

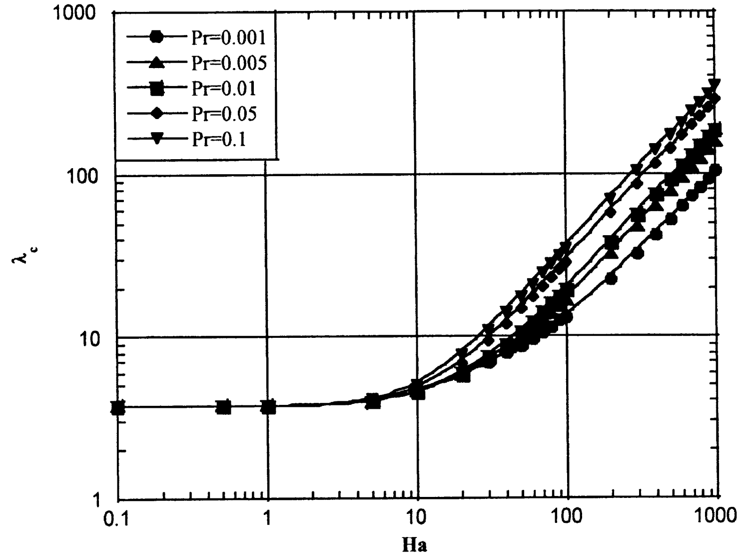

Fig. 4. Variation of the critical wavelength λ_c versus *Ha* for the oscillatory longitudinal instabilities under a horizontal transverse magnetic field (different values of the Prandtl number).

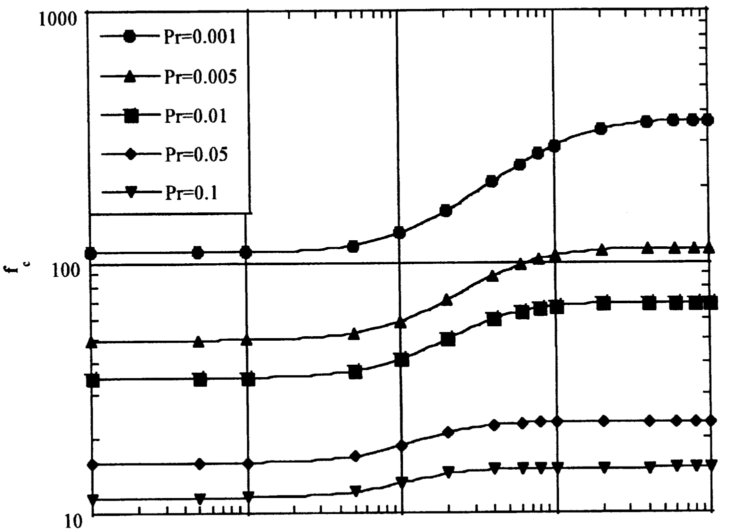

Fig. 5. Variation of the critical oscillation frequency f_c versus *Ha* for the oscillatory longitudinal instabilities under a horizontal transverse magnetic field (different values of the Prandtl number).

frequencies tend now towards asymptotic constant values at large *Ha*. These asymptotic values of frequency indicate a well-defined oscillatory regime in the high Hartmann number limit.

In the domain of small Prandtl numbers studied here ($0.001 \leqslant Pr \leqslant 0.1$), approximate power laws with respect to *Pr* can be found for λ_c and f_c in specific domains of *Ha*. In the domain of small *Ha*, as already shown in Ref. [1], they correspond to $\lambda_c \propto Pr^0$ and $f_c \propto Pr^{-1/2}$. In the domain of large Ha ($Ha > 100$), they are $\lambda_c \propto Pr^{1/4}$ and $f_c \propto Pr^{-3/4}$. As a consequence, in this small Pr and large Ha domain characteristic of the crystal growth situations controlled by magnetic field, the relevant scaling laws for both critical wavelengths and frequencies can be written as follows: $\lambda_c(Ha, Pr) \propto Ha\, Pr^{1/4}$ and $f_c(Ha, Pr) \propto Ha^0\, Pr^{-3/4}$.

As in the companion paper [1], it is possible to find the limit value Pr_l of Pr beyond which the longitudinal instability disappears. From the condition expressing that the denominator of Eq. (12) must be negative and using the fact that $\bar{\tau}_z/\bar{\upsilon}_z = -1/4\pi^2$ (thermally conducting boundaries) and $k \geqslant \pi$, we can find that $Pr_l = 3$ independently of Ha. Contrarily to the case of a vertical magnetic field [1], the horizontal transverse magnetic field does not change the potentially unstable domain which remains limited by $Pr_l = 3$, value already obtained by Gill [10] in the case without magnetic field. This result also means that for $Pr < 3$, there is no limitation in terms of Ha for the longitudinal instability to appear.

4. Linear stability analysis for the stationary transverse instabilities

We focus in this section on the stationary transverse instabilities defined by a zero wave number along y and corresponding to a two-dimensional mode which consists of rolls in the plane of the basic flow (xOz). The stabilization of these transverse instabilities by horizontal transverse or longitudinal magnetic field is considered. The stabilization is here only effective if the direction of the field is longitudinal to the basic flow ($\vec{e}_{B_0} = \vec{e}_x$). Indeed, it is now the transverse magnetic field ($\vec{e}_{B_0} = \vec{e}_y$) which is perpendicular to the plane of the transverse instabilities (xOz), and has then no effect on these instabilities (zero induced electric current). For this transverse field, the thresholds will then be independent of Ha and correspond to the values obtained without magnetic field ($Ha = 0$).

As for the vertical magnetic field [1], the analytical linear stability analysis relative to the transverse instabilities for a horizontal longitudinal magnetic field will be done in the small Prandtl number limit. As in Ref. [1], a single equation for the stream function φ in the (xOz) plane can be obtained from the linearized Navier–Stokes equations, namely:

$$\frac{\partial}{\partial t}\Delta\varphi + U_0(z)\big(\varphi_{xzz} + \varphi_{xxx}\big) - \frac{d^2 U_0(z)}{dz^2}\varphi_x = \Delta^2\varphi - Ha^2\varphi_{xx}. \tag{13}$$

Looking for solutions of Eq. (13) in the form

$$\varphi = e^{\sigma t} e^{ihx}\zeta(z) \tag{14}$$

and noting $U_0(z) = Gr\, \upsilon(z)$ and $\sigma = -ihc\, Gr$, one can obtain an ordinary differential equation of order four in z, namely

$$\frac{d^4\zeta}{dz^4} - \big(2h^2 + ih\, Gr(\upsilon(z) - c)\big)\frac{d^2\zeta}{dz^2} + \left(h^4 + h^2\,\mathrm{Ha}^2 + ih\, Gr\left(h^2(\upsilon(z) - c) + \frac{d^2\upsilon(z)}{dz^2}\right)\right)\zeta = 0. \tag{15}$$

Eq. (15) is solved with the same approach as in Ref. [1] (Taylor development at order 6), which allows to obtain the Grashof number $Gr(h, Ha)$ beyond which the transverse instabilities occur, as an explicit function of h and Ha, namely

$$Gr(h, Ha) = \sqrt{\frac{288N(h, Ha)}{h^2 D(h, Ha)}} \tag{16}$$

with

$$N(h, Ha) = h^{12} + A_{10}h^{10} + A_8h^8 + A_6h^6 + A_4h^4 + A_2h^2 + A_0,$$
$$D(h, Ha) = -h^6 + C_4h^4 + C_2h^2 + C_0. \tag{17}$$

The coefficients $(A_{2n})_{n=0..5}$ and $(C_{2n})_{n=0..2}$ are given in Appendix B. In order to find out the critical Grashof number beyond which instabilities occur, we have to determine the minimum of the $Gr(h, Ha)$ function with respect to h for different values of Ha. The results, given in Fig. 6, show that the critical Grashof number Gr_c remains almost constant for $Ha < 10$, then begins to increase with the increase of the Hartmann number, and finally reaches a clear asymptotic law, $Gr_c \propto Ha$, at high Ha. The variation of the

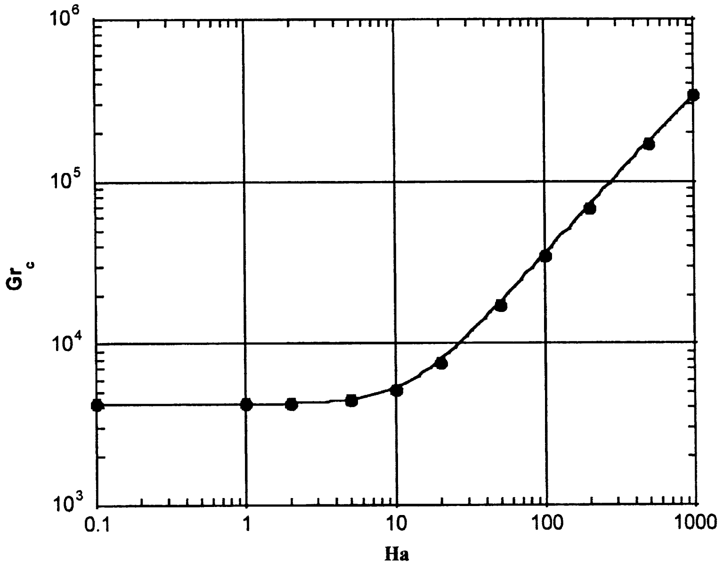

Fig. 6. Variation of Gr_c versus Ha for the transverse instabilities under a horizontal longitudinal magnetic field ($Pr = 0$).

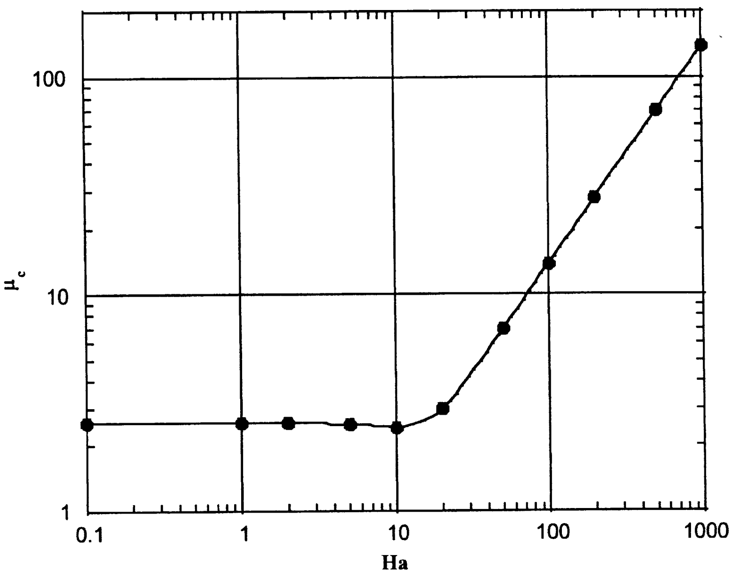

Fig. 7. Variation of the critical wavelength μ_c versus Ha for the transverse instabilities under a horizontal longitudinal magnetic field ($Pr = 0$).

critical wavelength $\mu_c = 2\pi/h_c$ versus the Hartmann number, which is illustrated in Fig. 7, shows a similar behaviour as for Gr_c. Indeed, μ_c shows a clear increase only for $Ha > 10$, and then reaches a similar asymptotic law, $\mu_c \propto Ha$, at high Ha.

Eq. (16) can also be used to see if there is a limit in terms of Ha for the transverse instabilities to appear. The numerator in Eq. (16) being strictly positive, a condition for the instabilities to occur is that $D(h, Ha)$ is positive. This condition gives $Ha^2 > g(h)$, where $g(h) = (h^6 + 72h^4 - 27648)/(h^4 + 24h^2)$ is negative for $h < 4.19$. In our case, $h_c < 2.6$ (asymptotic decrease of h_c for large Ha), which indicates that $g(h)$ is negative and that the above condition is then always verified. As a consequence, no Ha limit can be found for the transverse instabilities to occur under horizontal longitudinal magnetic field, contrarily to the case under vertical magnetic field where the stabilizing effect was stronger but limited on a small domain of Ha ($Ha < 11.471$).

5. Concluding remarks

The present study was devoted to the investigation of the effects of a constant horizontal transverse or longitudinal magnetic field on the development of the two types of instabilities susceptible to occur in a planar liquid metal layer heated through a horizontal temperature gradient. It is first to be noted that such horizontal magnetic fields do not affect the basic flow in the layer, contrarily to the case of the vertical magnetic field.

Concerning the stabilization effects, it is found that the magnetic field has no effect on the instabilities which develop in a plane perpendicular to the field, so that the horizontal transverse magnetic field only affects the oscillatory longitudinal instabilities, whereas the horizontal longitudinal magnetic field only affects the stationary transverse instabilities. In both cases, a similar stabilizing effect is obtained, with almost no variation of the thresholds for small Ha, then an increase which becomes significant around $Ha = 10$ before reaching an asymptotic linear variation, $Gr_c \propto Ha$, for large Ha. A similar variation with Ha is obtained in both cases for the wavelengths of the instabilities with also an asymptotic linear increase for large Ha. This variation of the wavelength corresponds to a rather classical behaviour: the increase of the convective cell length in the direction of the field. In the case of the oscillatory longitudinal instabilities, the frequency of the oscillations is also affected by the magnetic field, but the increase obtained for the intermediate values of Ha levels off and leads to asymptotic constant values of the frequency at large Ha, which indicates a clear asymptotic oscillatory state.

Compared to the vertical magnetic field for which Gr_c scaled as $\exp(Ha^2/21.6)$ (transverse instabilities) or Ha^2 (longitudinal instabilities) [1], the horizontal fields show less efficient stabilizing effects, but, contrarily to the vertical field, these effects are not limited in terms of Ha, so that quite important stabilizing effect can nevertheless be obtained at large Ha in the asymptotic domain already mentioned. We have also to point out that the horizontal fields increase the size of the marginal convective cells for both instabilities, whereas the vertical field increases the size of the transverse cells but reduces the size of the longitudinal cells. At last, the frequencies of the oscillatory longitudinal instabilities have been found to increase for horizontal magnetic field as well as for vertical magnetic field.

All the results presented in the present work and in the companion paper [1] have proved the importance of the magnetic field direction in the MHD stabilization process of convective flows: more significant stabilization with a vertical magnetic field, different stabilization effects with such a vertical field for the transverse or longitudinal instabilities, selective effect of the horizontal transverse or longitudinal fields which stabilize only one of these two instabilities. These two papers have also pointed out the main features of these stabilization processes and tried to provide characteristic scaling laws. This information could be of interest for the practician, especially in the crystal growth applications.

Appendix A

Value of $\bar{v}_z$:

$$\bar{v}_z = \frac{\int_{-1/2}^{1/2}(1/Gr)\frac{\mathrm{d}U_0}{\mathrm{d}z}\cos^2(\pi z)\,\mathrm{d}z}{\int_{-1/2}^{1/2}\cos^2(\pi z)\,\mathrm{d}z} = \frac{-1}{4\pi^2} \tag{A.1}$$

Value of $\bar{\tau}_z$ given for different boundary conditions:

- For thermally conducting boundaries:

$$\bar{\tau}_z = \frac{\int_{-1/2}^{1/2}(1/Gr\,Pr)\frac{\mathrm{d}T_0}{\mathrm{d}z}\cos^2(\pi z)\,\mathrm{d}z}{\int_{-1/2}^{1/2}\cos^2(\pi z)\,\mathrm{d}z} = \frac{1}{16\pi^4}, \tag{A.2}$$

- For thermally insulating boundaries:

$$\bar{\tau}_z = \frac{45+\pi^4}{720\pi^4}. \tag{A.3}$$

Appendix B

Values of the different coefficients $(A_{2n})_{n=0..5}$ and $(C_{2n})_{n=0..2}$:

$$A_{10} = 3\ Ha^2 + 288,$$
$$A_8 = 3\ Ha^4 + 576\ Ha^2 + 29,568,$$
$$A_6 = Ha^6 + 288\ Ha^4 + 36,096\ Ha^2 + 1,437,696,$$
$$A_4 = 6528\ Ha^4 + 995,328\ Ha^2 + 37,601,280,$$
$$A_2 = 11,059,200\ Ha^2 + 566,231,040,$$
$$A_0 = 4,246,732,800$$

and

$$C_4 = Ha^2 - 72,$$
$$C_2 = 24Ha^2,$$
$$C_0 = 27648.$$

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